

# Some remarks on topological full groups of Cantor minimal systems II

Hiroki Matui  
Graduate School of Science  
Chiba University  
Inage-ku, Chiba 263-8522, Japan

## Abstract

We prove that commutator subgroups of topological full groups arising from minimal subshifts have exponential growth. We also prove that the measurable full group associated to the countable, measure-preserving, ergodic and hyperfinite equivalence relation is topologically generated by two elements.

## 1 Introduction

We study algebraic properties of topological full groups of Cantor minimal systems. By a Cantor set, we mean a metrizable topological space which is compact, totally disconnected (the closed and open sets form a base for the topology) and has no isolated points. Any two such spaces are homeomorphic. A homeomorphism  $\varphi : X \rightarrow X$  is said to be minimal if for all  $x \in X$  the set  $\{\varphi^n(x) \mid n \in \mathbb{Z}\}$  is dense in  $X$ , or equivalently, there are no non-trivial closed  $\varphi$ -invariant subsets of  $X$ . A pair  $(X, \varphi)$  of a Cantor set  $X$  and a minimal homeomorphism  $\varphi$  of it is called a Cantor minimal system. The study of orbit structure of such dynamical systems was initiated by T. Giordano, I. F. Putnam and C. F. Skau in [5]. They classified Cantor minimal systems up to topological orbit equivalence. This classification was later extended to cover all minimal actions of finitely generated abelian groups on Cantor sets [4].

For a Cantor minimal system  $(X, \varphi)$ , the topological full group  $[[\varphi]]$  was introduced in [6]. The group  $[[\varphi]]$  consists of all homeomorphisms  $\psi : X \rightarrow X$  for which there exists a continuous map  $c : X \rightarrow \mathbb{Z}$  such that  $\psi(x) = \varphi^{c(x)}(x)$ . Clearly  $[[\varphi]]$  is infinite and countable. It was shown in [6, Corollary 4.4] that  $[[\varphi_1]]$  is isomorphic to  $[[\varphi_2]]$  as an abstract group if and only if  $\varphi_1$  is conjugate to  $\varphi_2$  or  $\varphi_2^{-1}$ . This result suggests that the algebraic structure of the topological full group  $[[\varphi]]$  is rich enough to recover the dynamics of  $\varphi$ . Since then various properties of  $[[\varphi]]$  have been studied in [10, 1, 7]. We collect below some of them. The commutator subgroup  $D([[ \varphi ]])$  of  $[[\varphi]]$  is simple ([10, Theorem 4.9] and [1, Theorem 3.4]). The quotient group  $[[\varphi]]/D([[ \varphi ]])$  is isomorphic to the direct sum of  $\mathbb{Z}$  and  $C(X, \mathbb{Z}/2\mathbb{Z})/\{f - f \circ \varphi \mid f \in C(X, \mathbb{Z}/2\mathbb{Z})\}$  ([6, Section 5] and [10, Theorem 4.8]), where  $C(X, \mathbb{Z}/2\mathbb{Z})$  denotes the  $\mathbb{Z}/2\mathbb{Z}$ -valued continuous functions on  $X$  with pointwise addition. The commutator subgroup  $D([[ \varphi ]])$  is finitely generated if and only if  $\varphi$  is a minimal subshift over a finite alphabet ([10, Theorem 5.4]). Recently, R. Grigorchuk and

K. Medynets [7] proved that  $[[\varphi]]$  is locally embeddable into finite groups. It is not yet known if  $[[\varphi]]$  is amenable.

In the present paper, we prove a couple of new results about  $[[\varphi]]$ . As mentioned above, for a minimal subshift  $\varphi$ ,  $D([[ \varphi ]])$  is finitely generated. It is then natural to consider the growth of  $D([[ \varphi ]])$ . We first observe that when  $\varphi$  is an odometer, any finitely generated subgroup of  $[[\varphi]]$  has polynomial growth (Proposition 2.1). Next, we prove that  $D([[ \varphi ]])$  contains the lamplighter group if and only if  $\varphi$  is not an odometer (Theorem 2.4). In particular, this implies that when  $\varphi$  is a minimal subshift,  $D([[ \varphi ]])$  has exponential growth (Corollary 2.5). Existence (or non-existence) of finitely generated subgroups of intermediate growth remains open. In Section 3, we discuss generators of the topological full group  $[[\varphi]]$  of Sturmian shifts  $\varphi$ . We have already shown in [10, Example 6.2] that  $[[\varphi]]$  is generated by three elements. Based on this result, J. Kittrell and T. Tsankov [9] proved that the measurable full group associated to the countable, measure-preserving, ergodic and hyperfinite equivalence relation on the standard probability space is topologically generated by at most three elements. In this paper we shall show that  $D([[ \varphi ]])$  is contained in a subgroup generated by two elements (Proposition 3.1). By using this, we can conclude that the measurable full group of the hyperfinite equivalence relation is topologically generated by two elements (Theorem 3.2). Also, this readily improves the estimates obtained in [9] for the number of topological generators of certain measurable full groups (Corollary 3.3).

## 2 Growth of topological full groups

In this section, we discuss growth of (finitely generated subgroups of) topological full groups. The reader may consult [8] for basic theory of growth of groups.

For  $m \in \mathbb{N}$ , we write  $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$  and identify it with  $\{0, 1, \dots, m-1\}$ . The cardinality of a set  $F$  is written  $|F|$ .

Let us first recall the odometers. Let  $(m_n)_n$  be a sequence of natural numbers such that  $m_n$  divides  $m_{n+1}$  and  $m_n \rightarrow \infty$  as  $n \rightarrow \infty$ . There exists a surjective homomorphism  $\rho_n : \mathbb{Z}_{m_{n+1}} \rightarrow \mathbb{Z}_{m_n}$  such that  $\rho_n(1) = 1$ . We let  $X$  be the inverse limit of  $\mathbb{Z}_{m_n}$  under the map  $\rho_n$ , that is,

$$X = \left\{ (x_n)_n \in \prod \mathbb{Z}_{m_n} \mid \rho_n(x_{n+1}) = x_n \right\}.$$

With the product topology,  $X$  is a Cantor set. Define a homeomorphism  $\varphi : X \rightarrow X$  by  $\varphi((x_n)_n) = (x_n + 1)_n$ . It is easy to see that  $(X, \varphi)$  is a Cantor minimal system. We call  $(X, \varphi)$  the odometer of type  $(m_n)_n$ .

**Proposition 2.1.** *Let  $(X, \varphi)$  be the odometer of type  $(m_n)_n$ . The topological full group  $[[\varphi]]$  is written as an increasing union of subgroups of the form  $\mathbb{Z}^{m_n} \rtimes S_{m_n}$ , where the symmetric group  $S_{m_n}$  acts on  $\mathbb{Z}^{m_n}$  by permutations of the coordinates. In particular, any finitely generated subgroup of  $[[\varphi]]$  has polynomial growth.*

*Proof.* Suppose that  $(X, \varphi)$  is the odometer of type  $(m_n)_n$ . For  $k \in \mathbb{N}$  and  $l \in \mathbb{Z}_{m_k}$ , we set  $U(k, l) = \{(x_n)_n \in X \mid x_k = l\}$ . Then  $\{U(k, l) \mid l \in \mathbb{Z}_{m_k}\}$  is a clopen partition of  $X$ . For  $\psi \in [[\varphi]]$ , we let  $c_\psi : X \rightarrow \mathbb{Z}$  be the continuous function satisfying  $\psi(x) = \varphi^{c_\psi(x)}(x)$ . Define a subgroup  $\Gamma_k \subset [[\varphi]]$  by

$$\Gamma_k = \{\psi \in [[\varphi]] \mid c_\psi \text{ is constant on } U(k, l) \text{ for each } l \in \mathbb{Z}_{m_k}\}.$$

Clearly  $\Gamma_k \subset \Gamma_{k+1}$  and the union of  $\Gamma_k$  equals  $[[\varphi]]$ .

Fix  $k \in \mathbb{N}$ . We would like to show that  $\Gamma_k$  is isomorphic to  $\mathbb{Z}^{m_k} \rtimes S_{m_k}$ . For any  $\psi \in \Gamma_k$ , there exists  $\tau \in S_{m_k}$  such that  $\psi(U(k, l)) = U(k, \tau(l))$ , and so we obtain a homomorphism  $\pi : \Gamma_k \rightarrow S_{m_k}$ . For each  $\tau \in S_{m_k}$ , one can define  $\psi \in \Gamma_k$  by  $\psi(x) = \varphi^{\tau(l)-l}(x)$  for  $x \in U(k, l)$ , where  $l$  and  $\tau(l)$  are regarded as elements in  $\{0, 1, \dots, m_k-1\}$ . The map  $\tau \mapsto \psi$  is a homomorphism from  $S_{m_k}$  to  $\Gamma_k$  and is a right inverse of  $\pi$ . If  $\psi \in \Gamma_k$  belongs to the kernel of  $\pi$ , then there exist  $n_l \in \mathbb{Z}$  for  $l \in \mathbb{Z}_{m_k}$  such that  $\psi(x) = \varphi^{n_l m_k}(x)$  holds for any  $x \in U(k, l)$ . Evidently,  $\psi \mapsto (n_l)_l$  gives an isomorphism from  $\text{Ker } \pi$  to  $\mathbb{Z}^{m_k}$ . Consequently,  $\Gamma_k$  is isomorphic to  $\mathbb{Z}^{m_k} \rtimes S_{m_k}$ .  $\square$

The following lemma is well known. For the convenience of the reader, we include an explicit proof.

**Lemma 2.2.** *Let  $(X, \varphi)$  be a Cantor minimal system. If  $(X, \varphi)$  is not an odometer, then there exists a continuous map  $\pi : X \rightarrow \{0, 1\}^{\mathbb{Z}}$  such that  $\pi(X)$  is infinite and  $\pi \circ \varphi = \sigma \circ \pi$ , where  $\sigma : \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$  is the shift.*

*Proof.* Let  $\{O_n \mid n \in \mathbb{N}\}$  be the set of all clopen subsets of  $X$ . For each  $n \in \mathbb{N}$ , we define a continuous map  $\pi_n : X \rightarrow \{0, 1\}^{\mathbb{Z}}$  by

$$\pi_n(x)_k = \begin{cases} 1 & \varphi^k(x) \in O_n \\ 0 & \varphi^k(x) \notin O_n \end{cases}$$

for  $x \in X$ . Then one has  $\pi_n \circ \varphi = \sigma \circ \pi_n$ .

Arguing by contradiction, we assume that  $\pi_n(X)$  is finite for all  $n \in \mathbb{N}$ . We will construct continuous maps  $\tilde{\pi}_n : X \rightarrow \mathbb{Z}_{m_n}$  and surjective homomorphisms  $\rho_n : \mathbb{Z}_{m_{n+1}} \rightarrow \mathbb{Z}_{m_n}$  such that  $\tilde{\pi}_n(\varphi(x)) = \tilde{\pi}_n(x) + 1$  for  $x \in X$ ,  $\tilde{\pi}_n = \rho_n \circ \tilde{\pi}_{n+1}$  and  $\pi_n$  factors through  $\tilde{\pi}_n$ . First, letting  $m_1 = |\pi_1(X)|$ , we can find a continuous map  $\tilde{\pi}_1 : X \rightarrow \mathbb{Z}_{m_1}$  such that  $\tilde{\pi}_1(\varphi(x)) = \tilde{\pi}_1(x) + 1$  and  $\pi_1$  factors through  $\tilde{\pi}_1$ . Suppose that we have constructed  $\tilde{\pi}_n : X \rightarrow \mathbb{Z}_{m_n}$ . Consider the continuous map  $\pi_{n+1} \times \tilde{\pi}_n : X \rightarrow \{0, 1\}^{\mathbb{Z}} \times \mathbb{Z}_{m_n}$  and let  $m_{n+1} = |(\pi_{n+1} \times \tilde{\pi}_n)(X)|$ . Then, identifying  $(\pi_{n+1} \times \tilde{\pi}_n)(X)$  with  $\mathbb{Z}_{m_{n+1}}$ , we can construct  $\tilde{\pi}_{n+1} : X \rightarrow \mathbb{Z}_{m_{n+1}}$  and  $\rho_n : \mathbb{Z}_{m_{n+1}} \rightarrow \mathbb{Z}_{m_n}$  as desired.

Let  $(Y, \psi)$  be the odometer of type  $(m_n)_n$ . Define the continuous map  $f : X \rightarrow Y$  by  $f(x) = (\tilde{\pi}_n(x))_n$ . Clearly we have  $f \circ \varphi = \psi \circ f$ . For any distinct points  $x, x' \in X$ , there exists  $O_n$  such that  $x \in O_n$  and  $x' \notin O_n$ , which means that  $f$  is injective. Thus  $(X, \varphi)$  is conjugate to  $(Y, \psi)$ , which completes the proof.  $\square$

In what follows, for a clopen subset  $O \subset X$ ,

$$1_O : X \rightarrow \mathbb{Z}_2$$

denotes the  $\mathbb{Z}_2$ -valued characteristic function of  $O$ . The following proposition is used in the proof of Theorem 2.4 in order to construct an embedding of the infinite direct sum of  $\mathbb{Z}_2$  into  $[[\varphi]]$ .

**Proposition 2.3.** *Let  $(X, \varphi)$  be a Cantor minimal system. If  $(X, \varphi)$  is not an odometer, then there exists a clopen subset  $O \subset X$  such that for any finite subset  $F \subset \mathbb{Z}$ , the function*

$$\sum_{k \in F} 1_O \circ \varphi^k$$

*is not identically zero mod 2.*

*Proof.* By the lemma above, there exists a continuous map  $\pi : X \rightarrow \{0, 1\}^{\mathbb{Z}}$  such that  $\pi(X)$  is infinite and  $\pi \circ \varphi = \sigma \circ \pi$ , where  $\sigma : \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$  is the shift. We identify  $\{0, 1\}$  with  $\mathbb{Z}/2\mathbb{Z}$ . Set

$$O = \{x \in X \mid \pi(x)_0 = 1\}.$$

In other words,  $1_O(x) = \pi(x)_0 \in \mathbb{Z}/2\mathbb{Z}$ . We would like to see that the clopen subset  $O \subset X$  satisfies the requirement. Let  $F \subset \mathbb{Z}$  be a finite subset. Suppose that the function

$$\sum_{k \in F} 1_O \circ \varphi^k$$

is identically zero mod 2. Put  $l = \min\{k \in F\}$  and  $m = \max\{k \in F\}$ . Assume that  $x, y \in X$  satisfies  $\pi(x)_n = \pi(y)_n$  for all  $n \in \{l, l+1, \dots, m\}$ . Then

$$\begin{aligned} \pi(x)_{l-1} &= 1_O(\varphi^{l-1}(x)) \\ &= 1_O(\varphi^{l-1}(x)) + \sum_{k \in F} 1_O(\varphi^{k-1}(x)) \\ &= \sum_{k \in F \setminus \{l\}} 1_O(\varphi^{k-1}(x)) \\ &= \sum_{k \in F \setminus \{l\}} \pi(x)_{k-1}, \end{aligned}$$

and so  $\pi(x)_{l-1} = \pi(y)_{l-1}$ . Repeating this procedure, we obtain  $\pi(x)_n = \pi(y)_n$  for every  $n \leq l$ . In the same way we get  $\pi(x)_n = \pi(y)_n$  for every  $n \geq m$ . Thus  $\pi(x) = \pi(y)$ . This means that the cardinality of  $\pi(X)$  is at most  $2^{m-l+1}$ , which is a contradiction.  $\square$

We call the wreath product

$$L = \left( \bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \right) \rtimes \mathbb{Z}$$

the lamplighter group, where the semi-direct product is taken with respect to the shift action. It is easy to see that  $L$  is finitely generated and that  $L$  contains a free semi-group on two generators. Hence  $L$  has exponential growth (see [8, VII.1] for example).

The technique we employ in the proof of the following theorem is essentially the same as that of [2, Theorem 8.1]. I am grateful to Koji Fujiwara for explaining this technique.

**Theorem 2.4.** *Let  $(X, \varphi)$  be a Cantor minimal system. The following three conditions are equivalent.*

- (1)  $(X, \varphi)$  is not an odometer.
- (2)  $D([\varphi])$  contains the lamplighter group  $L$ .
- (3)  $[[\varphi]]$  contains the lamplighter group  $L$ .

*Proof.* (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) immediately follows from Proposition 2.1. We show (1) $\Rightarrow$ (2). Choose a non-empty clopen subset  $U \subset X$  so that  $U, \varphi(U), \varphi^2(U)$  and  $\varphi^3(U)$  are disjoint. Let  $\psi$  be the first return map on  $U$  (see [5, Definition 1.5]). Letting  $\psi(x) = x$

for  $x \in X \setminus U$ , we may regard  $\psi$  as an element of  $[[\varphi]]$ . Define  $r = \psi \circ \varphi \circ \psi \circ \varphi^{-1}$ . Clearly  $r$  is of infinite order. For each clopen subset  $V \subset U$ , we define  $\tau_V \in [[\varphi]]$  by

$$\tau_V(x) = \begin{cases} \varphi(x) & x \in V \\ \varphi^{-1}(x) & x \in \varphi(V) \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that for any clopen subsets  $V, W \subset U$ ,  $\tau_V$  and  $\tau_W$  commute. Also, we have  $r \circ \tau_V \circ r^{-1} = \tau_{\psi(V)}$ . Furthermore, for clopen subsets  $V_1, V_2, \dots, V_n \subset U$ ,  $\tau_{V_1} \circ \tau_{V_2} \circ \dots \circ \tau_{V_n}$  equals the identity if and only if  $1_{V_1} + 1_{V_2} + \dots + 1_{V_n}$  equals zero (as a  $\mathbb{Z}_2$ -valued function). Since  $(X, \varphi)$  is not an odometer, neither is  $(U, \psi|_U)$ . It follows from the proposition above that there exists a clopen subset  $O \subset U$  such that for any finite subset  $F \subset \mathbb{Z}$ , the function

$$\sum_{k \in F} 1_O \circ \psi^k$$

is not identically zero mod 2. Define  $s = \tau_O$ . Because  $r^k \circ s \circ r^{-k} = \tau_{\psi^k(O)}$  for any  $k \in \mathbb{Z}$ , the homeomorphisms  $r^k \circ s \circ r^{-k}$  commute with each other. Moreover, for any non-empty finite subset  $\{k_1, k_2, \dots, k_n\} \subset \mathbb{Z}$ ,

$$(r^{k_1} \circ s \circ r^{-k_1}) \circ (r^{k_2} \circ s \circ r^{-k_2}) \circ \dots \circ (r^{k_n} \circ s \circ r^{-k_n})$$

is not the identity, because

$$1_{\psi^{k_1}(O)} + 1_{\psi^{k_2}(O)} + \dots + 1_{\psi^{k_n}(O)} \neq 0.$$

Therefore, the subgroup generated by  $r$  and  $s$  is isomorphic to the lamplighter group. The support of  $r$  and  $s$  is contained in  $U \cup \varphi(U)$ , and so the support of  $\varphi^2 \circ r \circ \varphi^{-2}$  and  $\varphi^2 \circ s \circ \varphi^{-2}$  is contained in  $\varphi^2(U) \cup \varphi^3(U)$ . Since  $U \cup \varphi(U)$  and  $\varphi^2(U) \cup \varphi^3(U)$  are disjoint, the subgroup

$$\langle r \circ \varphi^2 \circ r^{-1} \circ \varphi^{-2}, s \circ \varphi^2 \circ s^{-1} \circ \varphi^{-2} \rangle \subset D([[ \varphi ]])$$

is also isomorphic to the lamplighter group, which completes the proof.  $\square$

**Corollary 2.5.** *Let  $(X, \varphi)$  be a Cantor minimal system. If  $D([[ \varphi ]])$  is finitely generated, then it has exponential growth.*

*Proof.* By Proposition 2.1,  $\varphi$  is not an odometer. (We remark that  $D([[ \varphi ]])$  is finitely generated if and only if  $\varphi$  is a minimal subshift over a finite alphabet, see [10, Theorem 5.4].) It follows from the theorem above that  $D([[ \varphi ]])$  contains the lamplighter group  $L$ . As mentioned above,  $L$  has exponential growth. Therefore  $D([[ \varphi ]])$  has exponential growth, too.  $\square$

### 3 Generators of full groups

In this section, we will prove that the measurable full group associated to the countable, measure-preserving, ergodic and hyperfinite equivalence relation is topologically generated by two elements (Theorem 3.2).

### 3.1 Algebraic generators of topological full groups

Let  $\alpha \in (0, 1)$  be an irrational number and let  $(X, \varphi)$  be the Sturmian shift arising from the  $\alpha$ -rotation on  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . In [10, Example 6.2], it was shown that the topological full group  $[[\varphi]]$  is (algebraically) generated by the three elements  $\sigma_U$ ,  $\sigma_V$  and  $\varphi$ . In this subsection, for  $0 < \alpha < 1/6$ , we will show that the subgroup generated by  $\varphi$  and  $\sigma_U$  contains the commutator subgroup  $D([[ \varphi ]])$ .

Assume  $0 < \alpha < 1/6$ . We recall the notation used in [10, Example 6.2]. The clopen subset corresponding to the interval  $[0, \alpha) \subset \mathbb{T}$  is denoted by  $U \subset X$ . The element  $\sigma_U \in [[\varphi]]$  is defined by

$$\sigma_U(x) = \begin{cases} \varphi(x) & x \in U \\ \varphi^{-1}(U) & x \in \varphi(U) \\ x & \text{otherwise.} \end{cases}$$

**Proposition 3.1.** *In the setting above, let  $G \subset [[\varphi]]$  be the subgroup generated by  $\varphi$  and  $\sigma_U$ .*

- (1) *The commutator subgroup  $D([[ \varphi ]])$  is contained in  $G$ .*
- (2) *The subgroup  $G$  is normal and  $[[\varphi]]/G \cong \mathbb{Z}/2\mathbb{Z}$ .*

*Proof.* (1) Let  $\mathcal{W}$  be the set of all clopen subsets  $W \subset X$  such that  $\varphi^{-1}(W)$ ,  $W$ ,  $\varphi(W)$  are mutually disjoint. Clearly  $U$  is in  $\mathcal{W}$ . For  $W \in \mathcal{W}$ , we define  $\gamma_W \in [[\varphi]]$  by

$$\gamma_W(x) = \begin{cases} \varphi(x) & x \in \varphi^{-1}(W) \cup W \\ \varphi^{-2}(x) & x \in \varphi(W) \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that  $\gamma_W$  belongs to  $D([[ \varphi ]])$  (see the comment before [10, Lemma 5.2]). It was shown in [10, Lemma 5.2] that the commutator subgroup  $D([[ \varphi ]])$  is generated by  $\{\gamma_W \mid W \in \mathcal{W}\}$  (see also the comment before [10, Theorem 4.9]). For any  $n \in \mathbb{Z}$ ,  $\varphi^n(U)$  is the clopen subset corresponding to the interval  $[n\alpha, (n+1)\alpha) \subset \mathbb{T}$ . For any  $m, n \in \mathbb{Z}$ , if  $\varphi^m(U) \cap \varphi^n(U)$  is not empty, then it corresponds to either  $[m\alpha, (n+1)\alpha)$  or  $[n\alpha, (m+1)\alpha)$ . Hence any clopen subset  $W \subset X$  can be written as a finite disjoint union of clopen subsets of the form  $\varphi^m(U) \cap \varphi^n(U)$ . It follows that  $D([[ \varphi ]])$  is generated by

$$\{\gamma_W \mid \exists m, n \in \mathbb{Z}, W = \varphi^m(U) \cap \varphi^n(U)\}.$$

Since  $\varphi^n \circ \sigma_U \circ \varphi^{-n} = \sigma_{\varphi^n(U)}$ , one verifies

$$\begin{aligned} & (\varphi^n \circ \sigma_U \circ \varphi^{-n}) \circ (\varphi^{n-1} \circ \sigma_U \circ \varphi^{-n+1}) \circ (\varphi^n \circ \sigma_U \circ \varphi^{-n}) \circ (\varphi^{n-1} \circ \sigma_U \circ \varphi^{-n+1}) \\ &= \sigma_{\varphi^n(U)} \circ \sigma_{\varphi^{n-1}(U)} \circ \sigma_{\varphi^n(U)} \circ \sigma_{\varphi^{n-1}(U)} \\ &= \gamma_{\varphi^n(U)}, \end{aligned}$$

and so  $\gamma_{\varphi^n(U)}$  belongs to  $G$ . Suppose that the clopen subset  $\varphi^m(U) \cap \varphi^n(U)$  corresponds to the interval  $[m\alpha, (n+1)\alpha) \subset \mathbb{T}$ . Then  $\varphi^{n-2}(U)$ ,  $\varphi^{n-1}(U)$ ,  $\varphi^n(U) \cup \varphi^m(U)$ ,  $\varphi^{m+1}(U)$ ,

$\varphi^{m+2}(U)$  are mutually disjoint, because  $\alpha$  is less than  $1/6$ . Therefore, thanks to [10, Lemma 5.3 (ii)], we have

$$\gamma_{\varphi^{m+1}(U)} \circ \gamma_{\varphi^{n-1}(U)}^{-1} \circ \gamma_{\varphi^{m+1}(U)}^{-1} \circ \gamma_{\varphi^{n-1}(U)} = \gamma_{\varphi^m(U) \cap \varphi^n(U)},$$

and so  $\gamma_{\varphi^m(U) \cap \varphi^n(U)}$  belongs to  $G$ . When the clopen subset  $\varphi^m(U) \cap \varphi^n(U)$  corresponds to the interval  $[n\alpha, (m+1)\alpha) \subset \mathbb{T}$ , we obtain the same conclusion in a similar way. Hence  $G$  contains the commutator subgroup  $D([\varphi])$ .

(2) By [10, Example 5.2],  $[\varphi]$  is generated by  $\varphi$ ,  $\sigma_U$  and  $\sigma_V$ , where  $V \subset X$  is another clopen subset. The element  $\sigma_V$  is of order two and is not contained in  $G$ . We can check

$$\sigma_V \circ \varphi \circ \sigma_V = (\sigma_V \circ \varphi \circ \sigma_V \circ \varphi^{-1}) \circ \varphi \in G$$

and

$$\sigma_V \circ \sigma_U \circ \sigma_V = (\sigma_V \circ \sigma_U \circ \sigma_V \circ \sigma_U) \circ \sigma_U \in G.$$

It follows that  $G$  is normal and  $[\varphi]/G \cong \mathbb{Z}/2\mathbb{Z}$ .  $\square$

### 3.2 Topological generators of measurable full groups

In this subsection we follow the notation of [9]. Let  $X$  be a standard Borel space and let  $\mu$  be a non-atomic Borel probability measure on it. Denote by  $\text{Aut}(X, \mu)$  the group of all measure-preserving automorphisms of  $(X, \mu)$  (modulo null sets). We equip the group  $\text{Aut}(X, \mu)$  with the topology induced by the metric

$$d(f, g) = \mu(\{x \in X \mid f(x) \neq g(x)\})$$

and call it the uniform topology. For a countable, Borel, measure-preserving equivalence relation  $E$  on  $X$ , its measurable full group  $[E]$  is defined by

$$[E] = \{f \in \text{Aut}(X, \mu) \mid (x, f(x)) \in E \text{ for almost every } x \in X\}.$$

The measurable full group  $[E]$  is a closed subgroup of  $\text{Aut}(X, \mu)$  in the uniform topology and they turn out to be separable, and hence Polish. Following [9], we let  $t([E])$  denote the minimum number of topological generators of  $[E]$  (i.e. the minimum number of elements which generate a dense subgroup of  $[E]$ ). By using Proposition 3.1, we can improve some results for  $t([E])$  obtained in [9].

The following theorem answers [9, Question 4.3].

**Theorem 3.2.** *Let  $E$  be the countable, measure-preserving, ergodic and hyperfinite equivalence relation on the standard probability space  $(X, \mu)$ . Then  $t([E]) = 2$ .*

*Proof.* Let  $(X, \varphi)$  be the Sturmian shift arising from an irrational number  $\alpha \in (0, 1/6)$ . By Proposition 3.1, there exists  $\sigma_U \in [\varphi]$  such that the subgroup  $G \subset [\varphi]$  generated by  $\varphi$  and  $\sigma_U$  contains the commutator subgroup  $D([\varphi])$ .

Let  $E \subset X \times X$  be the equivalence relation induced by  $\varphi$ . There exists a unique  $\varphi$ -invariant Borel probability measure  $\mu$  on  $X$ . Then  $E$  is the countable, measure-preserving, ergodic and hyperfinite equivalence relation on the standard probability space  $(X, \mu)$ . By [9, Proposition 4.1],  $[\varphi]$  is dense in  $[E]$  in the uniform topology. In particular, the commutator subgroup  $D([\varphi])$  is dense in the commutator subgroup  $D([E])$ . It follows that  $G$  is dense in  $D([E])$ . Since  $[E]$  is simple by [3],  $[E] = D([E])$ . Therefore the group  $G$  generated by the two elements  $\varphi$  and  $\sigma_U$  is dense in  $[E]$ , which implies  $t([E]) = 2$ .  $\square$

The theorem above enables us to sharpen the estimates given in Theorem 4.10 and Corollary 4.12 of [9].

**Corollary 3.3.** *Let  $E$  be a countable, measure-preserving, ergodic equivalence relation on the standard probability space  $(X, \mu)$ .*

- (1) *If the cost of  $E$  is less than  $n$  for some  $n \in \mathbb{N}$ , then  $t([E]) \leq 2n$ .*
- (2) *If  $E$  is induced by a free action of the free group  $\mathbb{F}_n$ , then  $n + 1 \leq t([E]) \leq 2(n + 1)$ .*

**Acknowledgement.** The author would like to thank Koji Fujiwara and Konstantin Medynets for valuable discussions.

## References

- [1] S. Bezuglyi and K. Medynets, *Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems*, Colloq. Math. 110 (2008), 409–429. arXiv:math/0611173
- [2] F. Dahmani, K. Fujiwara and V. Guirardel, *Free groups of interval exchange transformations are rare*, preprint. arXiv:1101.5909
- [3] S. J. Eigen, *On the simplicity of the full group of ergodic transformations*, Israel J. Math. 40 (1981), 345–349.
- [4] T. Giordano, H. Matui, I. F. Putnam and C. F. Skau, *Orbit equivalence for Cantor minimal  $\mathbb{Z}^d$ -systems*, Invent. Math. 179 (2010), 119–158. arXiv:0810.3957
- [5] T. Giordano, I. F. Putnam and C. F. Skau, *Topological orbit equivalence and  $C^*$ -crossed products*, J. Reine Angew. Math. 469 (1995), 51–111.
- [6] T. Giordano, I. F. Putnam and C. F. Skau, *Full groups of Cantor minimal systems*, Israel J. Math. 111 (1999), 285–320.
- [7] R. Grigorchuk and K. Medynets, *Topological full groups are locally embeddable into finite groups*, preprint. arXiv:1105.0719
- [8] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000.
- [9] J. Kittrell and T. Tsankov, *Topological properties of full groups*, Ergodic Theory Dynam. Systems 30 (2010), 525–545.
- [10] H. Matui, *Some remarks on topological full groups of Cantor minimal systems*, Internat. J. Math. 17 (2006), 231–251. math.DS/0404117